13.1: Intro to 3D Vector Curves

To visualize 3D-curves, we start by

Step 1: Find surface/path of motion.

Step 2: Plot points.

2D Examples

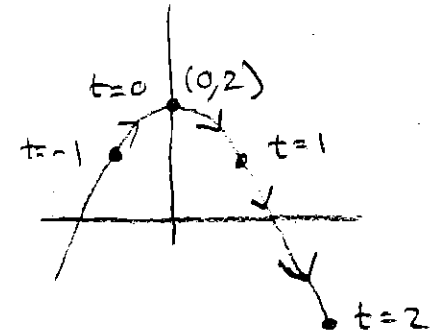
Eliminate the parameters

1. $x = t, y = 2 - t^2$

$$\Updownarrow$$

$$t = x \Rightarrow \boxed{y = 2 - x^2} \quad \text{PATH OF MOTION}$$

t	x	y
0	0	2
1	1	1
2	2	-2
-1	-1	1



2. $x = 3\cos(4t), y = 4\sin(4t)$

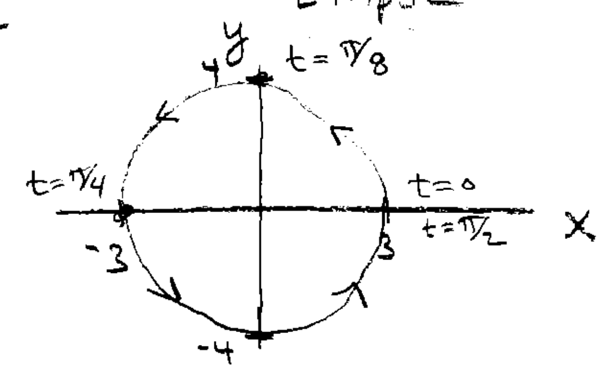
$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1 \quad \text{ALWAYS!}$$

NOTE: $\cos(4t) = \frac{x}{3}$ AND $\sin(4t) = \frac{y}{4}$

$$\Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \boxed{\frac{x^2}{9} + \frac{y^2}{16} = 1}$$

Ellipse

t	x	y
0	3	0
$\pi/8$	0	4
$\pi/4$	-3	0
$-\pi/8$	0	-4



3D Example

$$x = t, y = \cos(2t), z = \sin(2t)$$

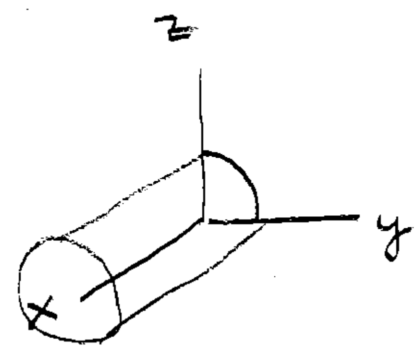
$\Updownarrow$

$t = x \implies$ $\left.\begin{array}{l} y = \cos(2x) \\ z = \sin(2x) \end{array}\right\}$ CURVE IS ON INTERSECTION OF THESE TWO SINUSOIDAL CYLINDERS] NOT VERY HELPFUL

$$(\cos(2x))^2 + (\sin(2x))^2 = 1$$ ALWAYS!

$\implies \boxed{y^2 + z^2 = 1}$ CIRCULAR CYLINDER IN X-DIRECTION

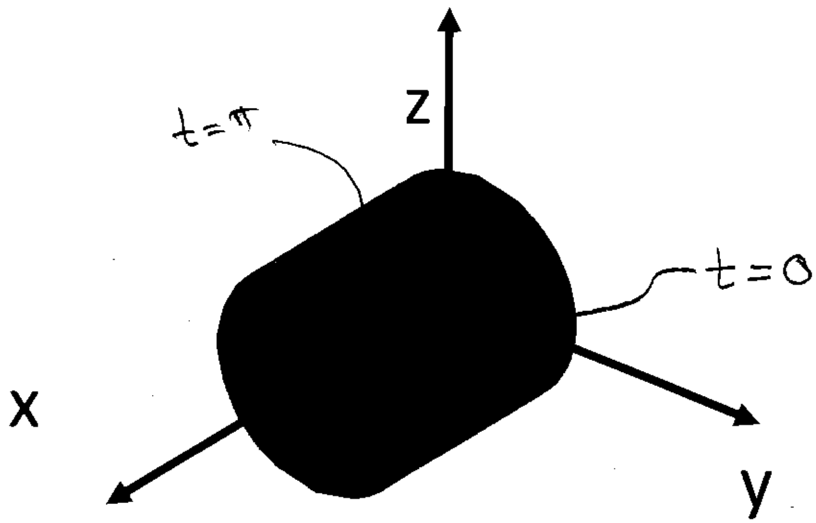
$\leftarrow$ SURFACE OF MOTION!



Example: All pts given by the equations

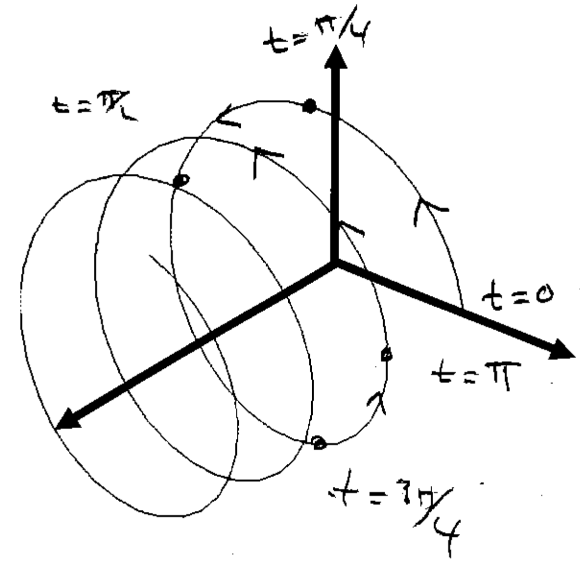
$$x = t\,, y = \cos(2t)\;\;, z = \sin(2t)$$

are on the cylinder: $y^2 + z^2 = 1.$



Now plot points!

t	x	y	z
0	0	1	0
π/4	π/4	0	1
π/2	π/2	−1	0
3π/4	3π/4	0	−1
π	π	1	0



Another 3D Examples

$$x = t\cos(t), y = t\sin(t)\ , z = t$$

$$\Updownarrow$$

$$t = z \Rightarrow \left.\begin{array}{l} x = z\cos(z) \\ \underbrace{y = z\sin(z)} \end{array}\right\}$$ ON INTERSECTION OF THESE CYLINDERS

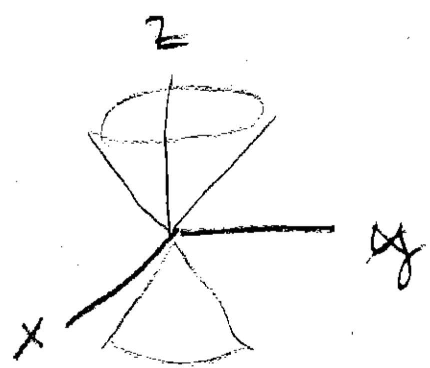
$$\cos(z) = \frac{x}{z}, \quad \sin(z) = y/z$$

$$(\cos(z))^2 + (\sin(z))^2 = 1 \quad \text{(ALWAYS)}$$

$$\Rightarrow \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1$$

$$\Rightarrow \frac{x^2}{z^2} + \frac{y^2}{z^2} = 1 \Rightarrow \boxed{x^2 + y^2 = z^2}$$

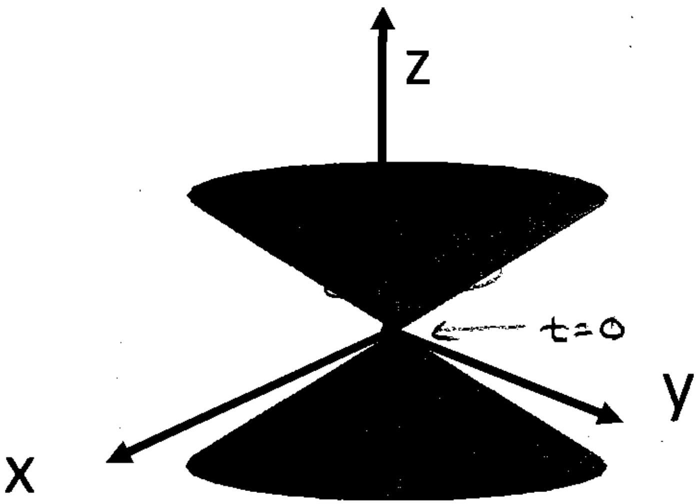
CONE! SURFACE OF MOTION



Example: All pts given by the equations

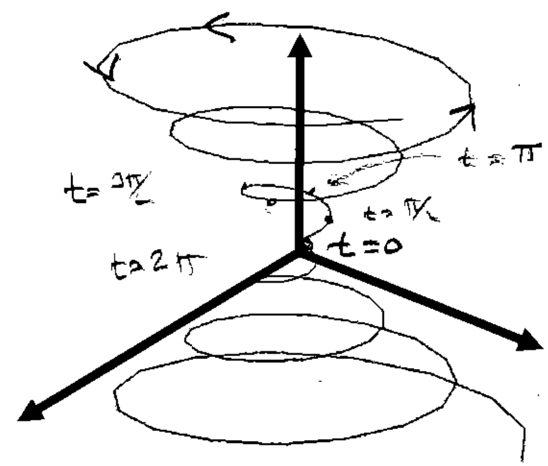
$$x = t\cos(t), y = t\sin(t)\ , z = t$$

are on the cone $z^2 = x^2 + y^2$.



Now plot points!

t	x	y	z
0	0	0	0
$\pi/2$	0	$\pi/2$	$\pi/2$
π	$-\pi$	0	π
$3\pi/2$	0	$-3\pi/2$	$3\pi/2$
2π	2π	0	2π



Intersection issues

For all intersection questions, combine the conditions

(a) ***Intersecting a curve and surface***.

Combine conditions

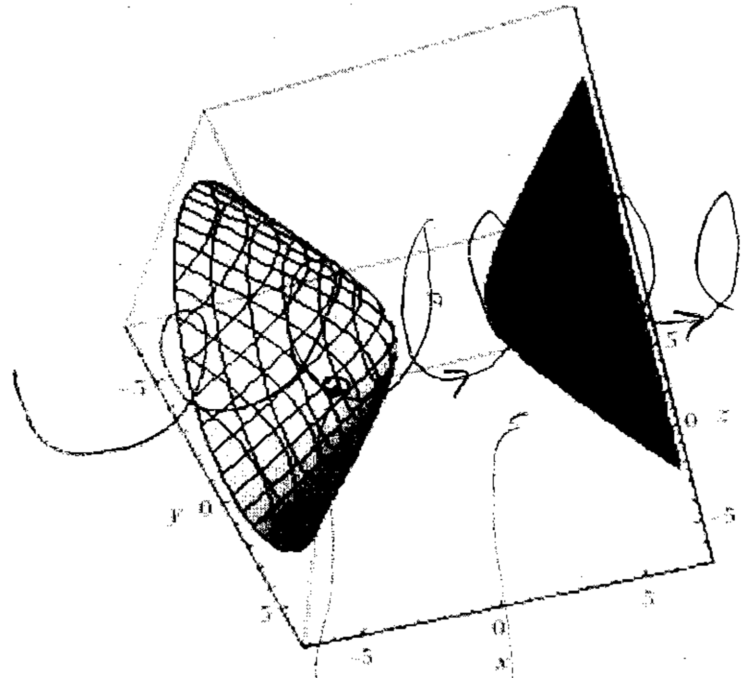
Example:

Find all intersections of

$$x = t, y = \cos(\pi\, t)\,, z = \sin(\pi\, t)$$

with the surface

$$x^2 - y^2 - z^2 = 3.$$



$$t^2 - \cos^2(\pi t) - \sin^2(\pi t) \stackrel{?}{=} 3$$

$$t^2 - (\underbrace{\cos^2(\pi t) + \sin^2(\pi t)}_{1}) = 3$$

$$t^2 - 1 = 3$$

$$t^2 = 4$$

$$t = \pm 2$$

$t = -2$

$(x, y, z) = (-2, 1, 0)$

$t = 2$

$(x, y, z) = (2, 1, 0)$

(b) ***Intersecting two curves.***

Use two different parameters!!!

Combine conditions.

We say the objects **collide** if the intersection happens at the same parameter value (i.e. same time).

Example:

Two particles are moving according to

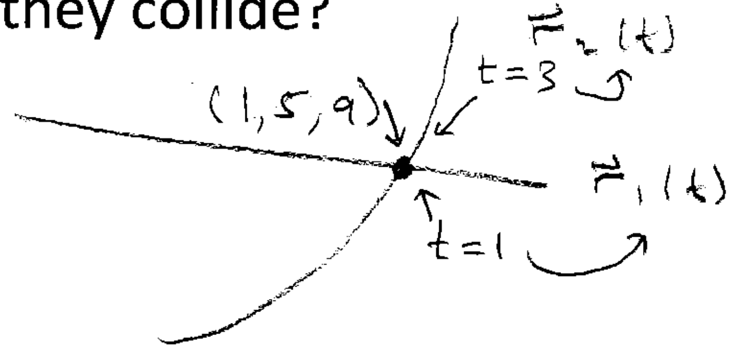
$$\boldsymbol{r}_1(t) = \langle t, 5t, 9\rangle, \text{and}$$

$$\boldsymbol{r}_2(t) = \langle t - 2, 5, t^2\rangle.$$

u

Do their paths intersect?

Do they collide?



$\boxed{1}$ $t = x \overset{?}{=} u - 2$

$\boxed{2}$ $5t = y \overset{?}{=} 5 \Leftrightarrow t = 1$

$\boxed{3}$ $9 = z \overset{?}{=} u^2 \Leftrightarrow u = \pm 3$

NEED <u>ALL THREE EQUAL</u>!

$t=1, u=-3$ DOES <u>NOT</u> WORK FOR ALL THREE

$1 = x \neq (-3) - 2$

$t=1, u=3$ DOES WORK FOR ALL THREE

$1 = x = 3 - 2$ ✓

$5(1) = y = 5$ ✓

$9 = z = (3)^2$ ✓

THE PATHS INTERSECT AT $\boxed{(1,5,9)}$

WHEN $t=1$ FOR $\vec{r}_1(t)$ AND

$u = t = 3$ FOR $\vec{r}_2(t)$

THEY DO <u>NOT</u> COLLIDE THEY REACH THIS POINT AT DIFFERENT TIMES.

(c) ***Intersecting two surfaces*.**

Answer will be a 3D curve.

To parameterize the curve:

Let one variable be t. Solve for others in terms of t.

OR

For circle/ellipse try

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow \begin{aligned} x &= \mathrm{a}\cos(t) \\ y &= \mathrm{b}\sin(t) \end{aligned}$$

Examples

1. Find *any* parametric equations that describe the curve of intersection of

$$z = 2x + y^2 \text{ and } \underbrace{z = 2y}$$

ALL IN TERMS OF y

Let $y = t$ (OR ANY OTHER FUNCTION)

$\Rightarrow z = 2t$

$\Rightarrow 2t = 2x + t^2$

So $2x = 2t - t^2$

$x = t - \frac{1}{2}t^2$

$$\boxed{\begin{aligned} x &= t - \tfrac{1}{2}t^2 \\ y &= t \\ z &= 2t \end{aligned}}$$

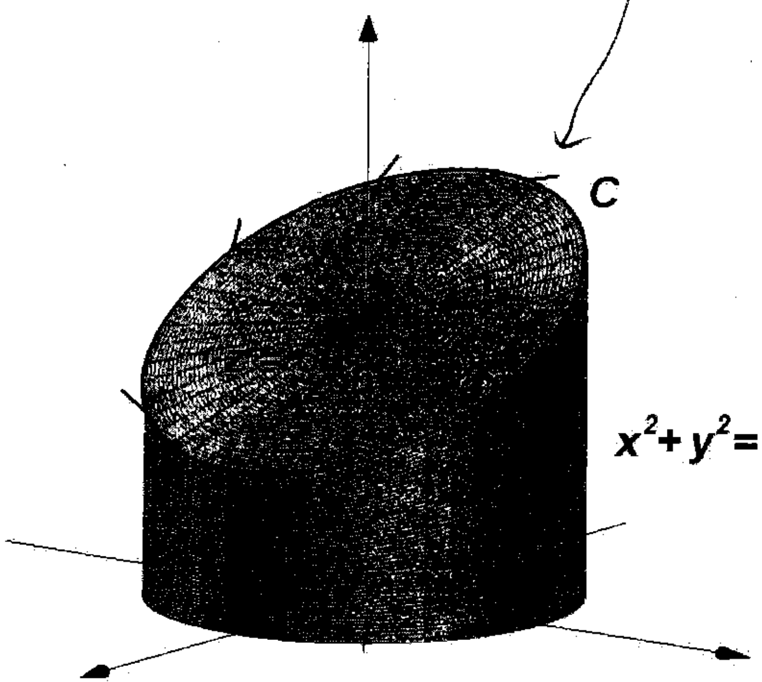
ONE PARAMETERIZATION FOR CURVE OF INTERSECTION

2. Find *any* parametric equations that describe the curve of intersection of

$$\underbrace{x^2 + y^2 = 1}\ \text{and}\ z = 5 - x$$

$$\boxed{\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \\ \Rightarrow z &= 5 - \cos(t) \end{aligned}}$$

ONE PARAMETERIZATION



3. Find *any* parametric equations that describe the curves of intersection of

$$x^2 + y^2 + z^2 = 1 \text{ and } z^2 = x^2 + y^2$$

SPHERE CONE

COMBINE FIRST

$$x^2+y^2+(x^2+y^2)=1$$

$$\Rightarrow 2x^2+2y^2=1$$

$$\Rightarrow x^2+y^2=\frac{1}{2}$$

CIRCLE

$$x = \frac{1}{\sqrt{2}}\cos(t)$$

$$y = \frac{1}{\sqrt{2}}\sin(t)$$

$$z^2 = \frac{1}{2} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$$

TWO CURVES

$$\boxed{\begin{aligned} x &= \frac{1}{\sqrt{2}}\cos(t) \\ y &= \frac{1}{\sqrt{2}}\sin(t) \\ z &= \pm\frac{1}{\sqrt{2}} \end{aligned}}$$